ROTATION

We recall that the most general rotation of space is given by three angles $\alpha, \beta, \gamma$. This rotation which we shall denote by $R(\alpha, \beta, \gamma)$ transforms the three basic coordinate vectors 

$$e_1 = (1, 0, 0), \quad f_1 = (0, 1, 0), \quad g_1 = (0, 0, 1)$$

by three successive rotations of angles $\alpha, \beta, \gamma$ according to the following procedure

1. First a rotation of angle $\alpha$ around $g_1$ keeping $g_1$ fixed, yielding three new coordinate vectors
   $$e_2 = \cos \alpha \ e_1 + \sin \alpha \ f_1, \quad f_2 = -\sin \alpha \ e_1 + \cos \alpha \ f_1, \quad g_2 = g_1(\alpha)$$

2. Then a rotation of angle $\beta$ around $e_2$ keeping $e_2$ fixed, yielding three new coordinate vectors
   $$e_3 = e_2, \quad f_3 = \cos \beta \ f_2 + \sin \beta \ g_2, \quad g_3 = -\sin \beta \ f_2 + \cos \beta \ g_2$$

3. Finally a rotation of angle $\gamma$ around $g_3$ keeping $g_3$ fixed, yielding three new coordinate vectors
   $$e_4 = \cos \alpha \ e_3 + \sin \alpha \ f_3, \quad f_4 = -\sin \alpha \ e_3 + \cos \alpha \ f_3, \quad g_4 = g_3(\gamma)$$

Carrying out the substitutions given in $(\alpha)$, $(\beta)$ and $(\gamma)$, a simple computation (that may be checked with MATHEMATICA), yields that

$$R(\alpha, \beta, \gamma) e_1 = (\cos \gamma \cos \alpha - \sin \alpha \cos \beta \sin \gamma, \sin \alpha \cos \gamma + \cos \alpha \cos \beta \sin \gamma, \sin \beta \sin \gamma)$$

$$R(\alpha, \beta, \gamma) f_1 = (-\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma, -\sin \alpha \sin \gamma + \cos \alpha \cos \beta \cos \gamma, \sin \beta \cos \gamma)$$

$$R(\alpha, \beta, \gamma) g_1 = (\sin \alpha \sin \beta, -\cos \alpha \sin \beta, \cos \beta)$$

If we take a point $P$ in space with coordinates $(x, y, z)$ its image by the rotation $R(\alpha, \beta, \gamma)$, is by definition, the point

$$P' = (x', y', z') = R(\alpha, \beta, \gamma) \ P$$

whose coordinates are given by the equation

$$R(\alpha, \beta, \gamma) P = x \ R(\alpha, \beta, \gamma) \ e_1 + y \ R(\alpha, \beta, \gamma) \ f_1 + z \ R(\alpha, \beta, \gamma) \ g_1 .$$

(1)

For convenience, let $R = \|R[i; j]\|$ be the matrix

$$R = \begin{pmatrix}
\cos \gamma \cos \alpha - \sin \alpha \cos \beta \sin \gamma & -\sin \gamma + \cos \alpha \cos \beta \sin \gamma & \sin \beta \sin \gamma \\
\sin \alpha \cos \gamma + \cos \alpha \cos \beta \sin \gamma & -\cos \gamma \cos \alpha - \sin \alpha \cos \beta \sin \gamma & \sin \beta \cos \gamma \\
\sin \beta \sin \gamma & -\cos \beta \sin \gamma & \cos \beta \\
\end{pmatrix}$$

This given, it is not difficult to show that (1) is equivalent to the equations

$$x' = R_{11} x + R_{12} y + R_{13} z$$

$$y' = R_{21} x + R_{22} y + R_{23} z$$

$$z' = R_{31} x + R_{32} y + R_{33} z$$

(2)
Note that, using matrix notation, these equations can simply be written in the form $v' = Rv$ where $v$ and $v'$ respectively denote the column vectors with components $x, y, z$ and $x', y', z'$.

The rotation matrix we have presented above is customarily used in Astronomy. Its advantage lies in that it contains the smallest number of parameters. But it has the disadvantage of not exhibiting the fundamental fact that every rigid motion of space which has a fixed point has also a fixed line. This means that the motion can always be achieved by a single rotation around a fixed line.

We shall present next another way of moving objects, which is based on this principle. You may use this alternate way in carrying out the assignment.

The idea is to choose first a vector $D$ giving the direction and orientation of the fixed line $L$. Then choose the angle $\alpha$ by which we wish to rotate. In the adjacent display we exhibit $D$ as a dark line, a given point $P$ by a large disk and, by smaller disks, the successive images of $P$ as we carry 5 successive rotations of angle $2\pi/8$.

Suppose that $D = \{A, B, C\}$ and we want to rotate the point $P = \{x, y, z\}$ around $L$ by the angle $\alpha$, then the equation giving the point $P' = \{x', y', z'\}$ which is the image of $P$ under this rotation, can be obtained as follows. To begin, we compute the unit vector $d = \{u, v, w\}$ in the direction of $L$ by setting

\[
  u = \frac{A}{\sqrt{A^2 + B^2 + C^2}}, \quad v = \frac{B}{\sqrt{A^2 + B^2 + C^2}}, \quad w = \frac{C}{\sqrt{A^2 + B^2 + C^2}},
\]

Next, we compute the auxiliary vectors:

\[
  P_1 = (d, P) \ d \quad \text{(projection of $P$ along $L$)}
\]
\[
  P_2 = P - P_1 \quad \text{(the portion of $P$ perpendicular to $L$)}
\]
\[
  N = d \times P_2 \quad \text{(the vector obtained by rotating $P_2$ by 90° around $L$)}
\]

This done we simply have

\[
  P' = P_1 + \cos[\alpha] P_2 + \sin[\alpha] N.
\]

To implement this write a procedure with heading

\[
  \text{ROTPT}[P, D, \alpha]
\]

which returns $P'$ as indicated above, taking account that $(d, P)$ and $d \times P_2$ denote “scalar” and “cross” products respectively. More precisely,

\[
  (d, P) = ux + vy + wz
\]

and, if $P_2 = \{x_2, y_2, z_2\}$ then

\[
  d \times P_2 = \{ vz_2 - wy_2, wx_2 - uz_2, uy_2 - vx_2 \}
\]

\[\text{\(\uparrow\) Counterclockwise as seen by a person with feet at the origin of D and head on the tip of D}\]