

**ECE 287A Presentation**

**Capacity, Diversity and Multiplexing Gain  
for MIMO channels**

**Yuzhe Jin**

**June 12, 2007**

# Outline

---

- Background
- Capacity of MIMO channels
  - Deterministic
  - Fading
  - Outage
- Diversity and Multiplexing Gains
  - Motivation
  - Fundamental tradeoff
  - Two practical schemes
- Extension to Multiple-Access Channels
- Conclusion

# Background: SISO Channel

---

Before we start, let's recall the discrete-time AWGN channel:

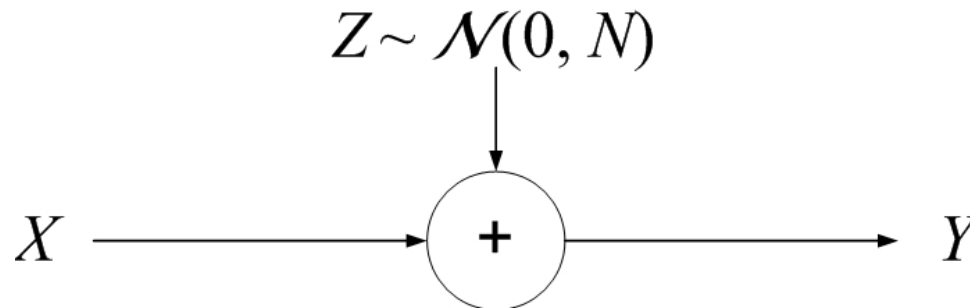


Figure 1: AWGN Channel

Some important observations:

- A power constraint  $P$  on input  $X$
- Capacity

$$C = \max_{p(x), P} I(X; Y) = \log \left( 1 + \frac{P}{N} \right)$$

# Background: Multiple Antennas

---

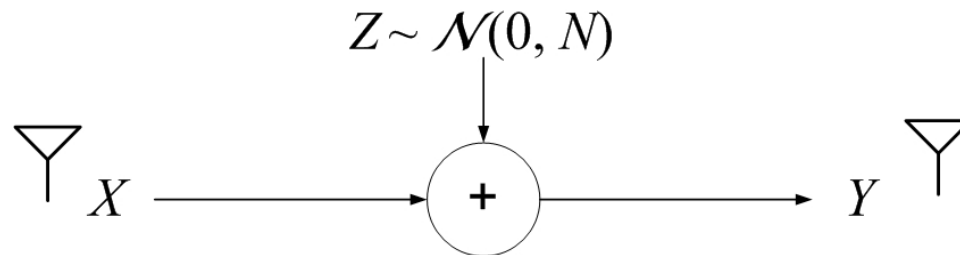


Figure 2: Single-input single-output

What if we have multiple antennas at each side?

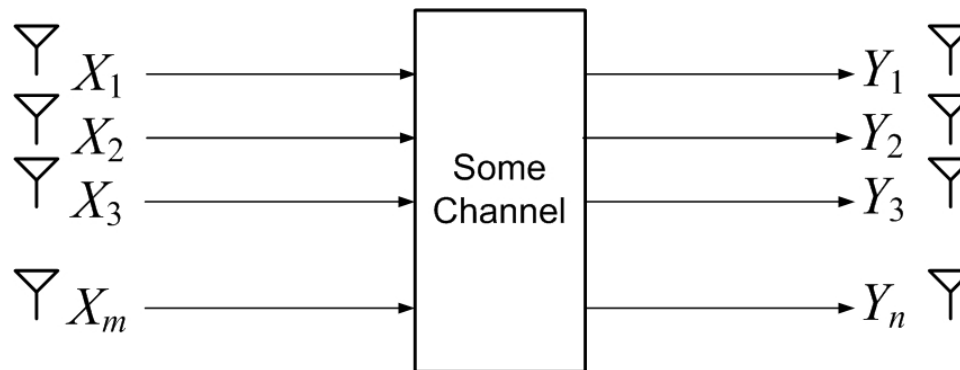


Figure 3: Multiple-input multiple-output

# Background: Multiple-Antenna Gaussian Channels

---

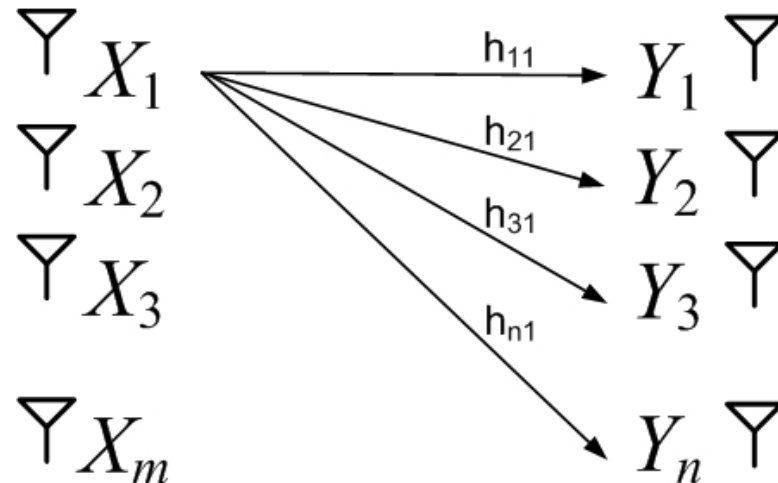


Figure 4: Pair-wise relation

Adding in Gaussian noise, we have the channel model:

$$Y = HX + N$$

where  $X_{m \times 1}$  is the channel input,  $Y_{n \times 1}$  is channel output.  $H_{n \times m}$  is the coefficient associated with each path between a transmit antenna and a receiver antenna,  $N_{n \times 1}$  is zero-mean complex Gaussian noise, and  $E(NN^H) = I$ .

## Capacity: When $H$ is fixed and known

---

The channel is  $Y = HX + N$ . Suppose  $E(XX^H) = Q$ .

$$\begin{aligned} I(X; Y) &= H(Y) - H(Y|X) \\ &= H(Y) - H(N) \\ &= \log \det(I + HQH^H) \\ &= \log \det(I + QH^H H) && \text{(Property of det)} \\ &= \log \det(I + \Lambda^{1/2} U Q U^H \Lambda^{1/2}) && \text{(eigen-decomposition)} \\ &= \log \det(I + \Lambda^{1/2} \tilde{Q} \Lambda^{1/2}) \\ &\leq \prod_i (1 + \tilde{Q}_{ii} \lambda_i) \end{aligned}$$

where  $\tilde{Q} = U Q U^H$ , and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$ .

## Capacity: When $H$ is fixed and known, Cont'd

---

$$I(X; Y) \leq \prod_i (1 + \tilde{Q}_{ii} \lambda_i)$$

With equality, when  $\tilde{Q}$  is diagonal, and optimal diagonal entries can be found via water-filling:

$$\tilde{Q}_{ii} = (\mu - \lambda_i^{-1})^+, \quad i = 1, 2, 3, \dots, m$$

where  $\mu$  is chosen to satisfy  $\sum_i \tilde{Q}_{ii} = P$ .

## Capacity: When $H$ is random

---

- Suppose each entry of  $H$  is independently drawn from a zero-mean complex Gaussian distribution, with independent real and imaginary parts, each with variance  $1/2$ .
- Gaussian Channel with Rayleigh Fading
- What is the capacity?

## Capacity: When $H$ is random

---

Assumption:  $H$  is known at receiver, but not known at transmitter.

$$I(X; \underbrace{(Y, \mathbf{H})}_{\text{channel output}}) = E_{\mathbf{H}}[\log \det(I + \mathbf{H}\mathbf{Q}\mathbf{H}^H)]$$

It's shown that the optimal input should be a circular symmetric complex Gaussian zero-mean and covariance  $(P/m)I$ . Then,

$$I(X; (Y, \mathbf{H})) = E_{\mathbf{H}}[\log \det(I + \frac{P}{m}\mathbf{H}\mathbf{H}^H)]$$

Let  $\mathbf{W} = \mathbf{H}\mathbf{H}^H, m \leq n; \mathbf{H}^H\mathbf{H}, m > n$ .  $\mathbf{W}$  is Wishart Distributed, we can represent the mutual information in terms of the eigenvalues of  $\mathbf{W}$ .

$$I(X; (Y, \mathbf{H})) = \min(m, n) \cdot E[\log(1 + \frac{P}{m}\lambda_1)]$$

Observation: MIMO can be interpreted as multiple parallel spatial channels.

# Capacity: When $H$ is random, but realized only once

---

- Ergodic capacity doesn't apply to this case. There will always be a positive probability that the realization of  $H$  is too bad to support a target rate  $R$ .
- Definition of Outage  
Outage is defined as the event that the mutual information of the channel does not support a target rate, due to a “bad” realization of channel coefficients  $H$ .
- Outage Probability: The probability that an outage event occurs.

$$P(\underbrace{R}_{\text{desired rate}} > \underbrace{I(X, (Y, H))}_{\text{instantaneous mutual information}}) = \epsilon$$

- Question:
  - Given a target rate, what's the outage probability?
  - Given an outage probability, what's the max rate we can achieve?

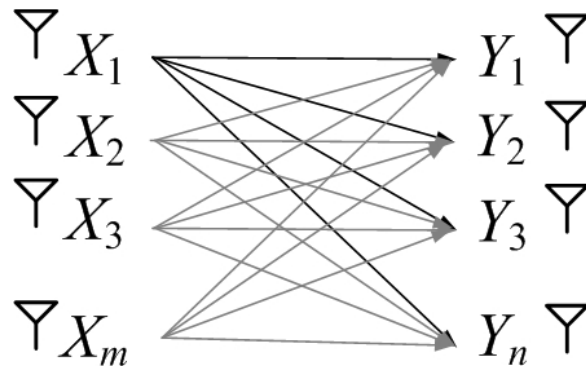
## Capacity: For more complicated cases

---

- When receiver doesn't know the channel, what is the capacity and corresponding input? and other complicated channel models?
- For more complicated channels, we may not be able to figure out channel capacity in closed form. Meanwhile, the capacity-achieving input distribution is as well hard to characterize.
- Is there other metric that captures the characteristic of a channel, other than capacity?
- In practice, given a coding scheme, is there other metric that can give us more insight about the performance?
- We may need more than capacity result.

# Diversity

---



In a MIMO Channel with  $m$  transmit antennas and  $n$  receive antennas,

- Each pair of transmit-receive antennas provides a signal path from the transmitter to receiver.
- If we send signals carrying the same information through different paths, multiple independently faded replicas of the data symbol can be obtained at the receiver.
- More reliable reception can be achieved. (“reliable”?)

## Diversity: Reliable Reception

---

Suppose we have a slow Rayleigh fading channel with a single transmitter and  $n$  receivers,

- $n$  different fading paths
- The average error probability can be made to decay like  $\frac{1}{\text{SNR}^n}$ , at high SNR. Note for a single-antenna fading channel, it's  $\frac{1}{\text{SNR}}$
- Use multiple paths to *combat* fading.
- A question: Can fading be beneficial?

# Multiplexing

---

- If the path gains between individual transmit-receive antenna pairs fade independently, the channel matrix will be well-conditioned with high probability.
- Higher rate. (MIMO as multiple parallel spatial channels)
- It's shown that at high SNR regime, the channel capacity is about

$$C(\text{SNR}) = \min(m, n) \log \text{SNR} + O(1)$$

- The channel capacity is increased by using multiple antennas due to spacial multiplexing. We can transmit independent information streams in parallel through the spacial channels.

# Can we get both?

---

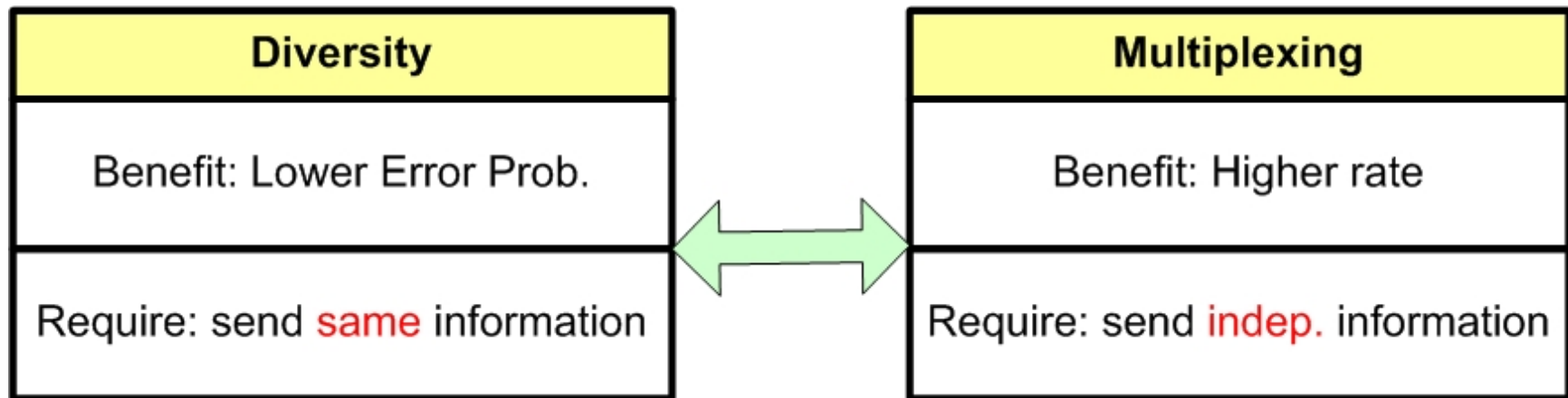


Figure 5: A conflict?

Obviously, the conflict between the two suggests a fundamental tradeoff between benefits obtained from diversity and multiplexing.

## Definitions of Diversity and Multiplexing Gain [2]

---

Consider a family of codes  $C(\text{SNR})$  of block length  $l$ , one at each SNR level. Let  $R(\text{SNR})$  be the rate of the code  $C(\text{SNR})$ .

- $C(\text{SNR})$  is said to achieve diversity gain  $d$  if the average error probability

$$\lim_{\text{SNR} \rightarrow \infty} \frac{P_e(\text{SNR})}{\log \text{SNR}} = -d$$

- $C(\text{SNR})$  is said to achieve the multiplexing gain  $r$  if the data rate

$$\lim_{\text{SNR} \rightarrow \infty} \frac{R(\text{SNR})}{\log \text{SNR}} = r$$

- For each  $r$ , define  $d^*(r)$  to be the supremum of the diversity advantage achieved over all schemes.

## Optimal Tradeoff between $d$ and $r$ [2]

*Theorem:* Assume  $l \geq m + n - 1$ . The optimal tradeoff curve  $d^*(r)$  is given by the piecewise-linear function connecting the points  $(k, d^*(k))$ ,  $k = 0, 1, \dots, \min(m, n)$ , where  $d^*(k) = (m - k)(n - k)$ . In particular,  $d_{\max}^* = mn$ ,  $r_{\max}^* = \min(m, n)$ .

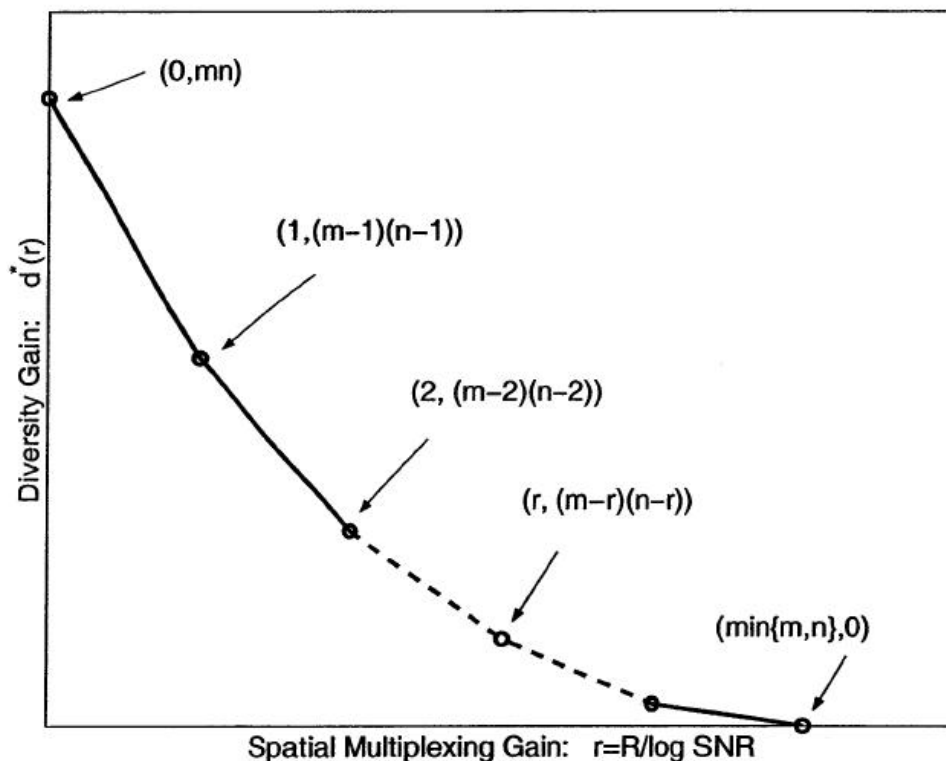
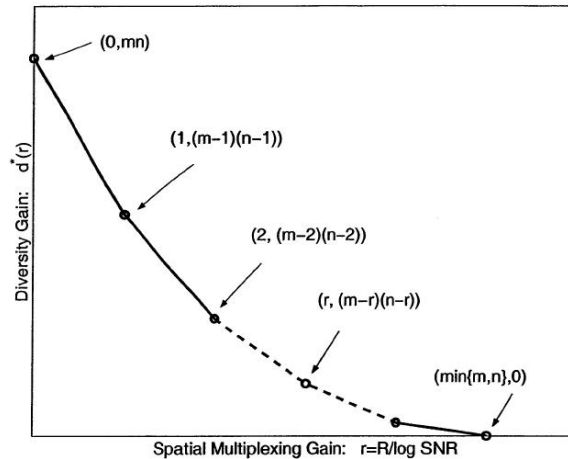


Figure 6: Diversity-multiplexing tradeoff, for general  $m, n, l \geq m + n - 1$

# How do we interpret the Optimal Tradeoff?

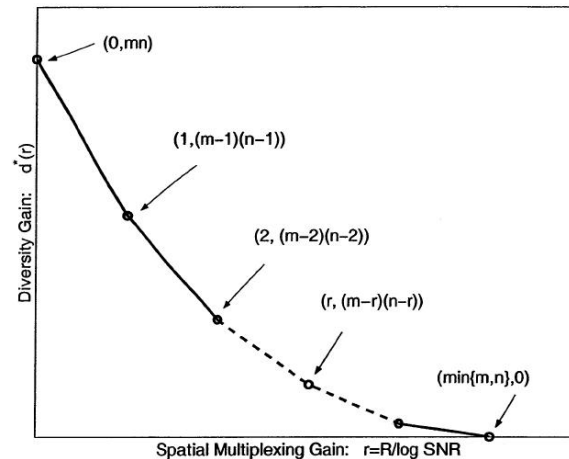
---



- The optimal tradeoff curve intersects the  $r$  axis at  $\min(m, n)$ , i.e. the maximum achievable spatial multiplexing gain  $r_{\max}^*$  is the total number of degrees of freedom provided by the channel as suggested by the ergodic capacity result.
- Intuitively, as  $r \rightarrow r_{\max}^*$ , the data rate approaches the ergodic capacity.
- At this point, however, no positive diversity gain can be achieved, i.e. no protection against the randomness in the fading channel.

# How do we interpret the Optimal Tradeoff?

---



- The curve intersects the  $d$  axis at the maximal diversity gain  $d_{\max}^* = mn$ , corresponding to the total number of random fading coefficients that a scheme can average over.
- In order to achieve the maximal diversity gain, no positive spatial multiplexing gain can be obtained at the same time.

# How do we interpret the Optimal Tradeoff?

---

- Bridges the gap between the two design criteria
- Positive diversity gain and multiplexing gain can be achieved simultaneously.
- Increasing one comes at a price of decreasing the other.
- This tradeoff curve provides a more complete picture of the achievable performance over multiple-antenna channels than two extreme points corresponding to maximum diversity/multiplexing gain.

# How do we interpret the Optimal Tradeoff?

---

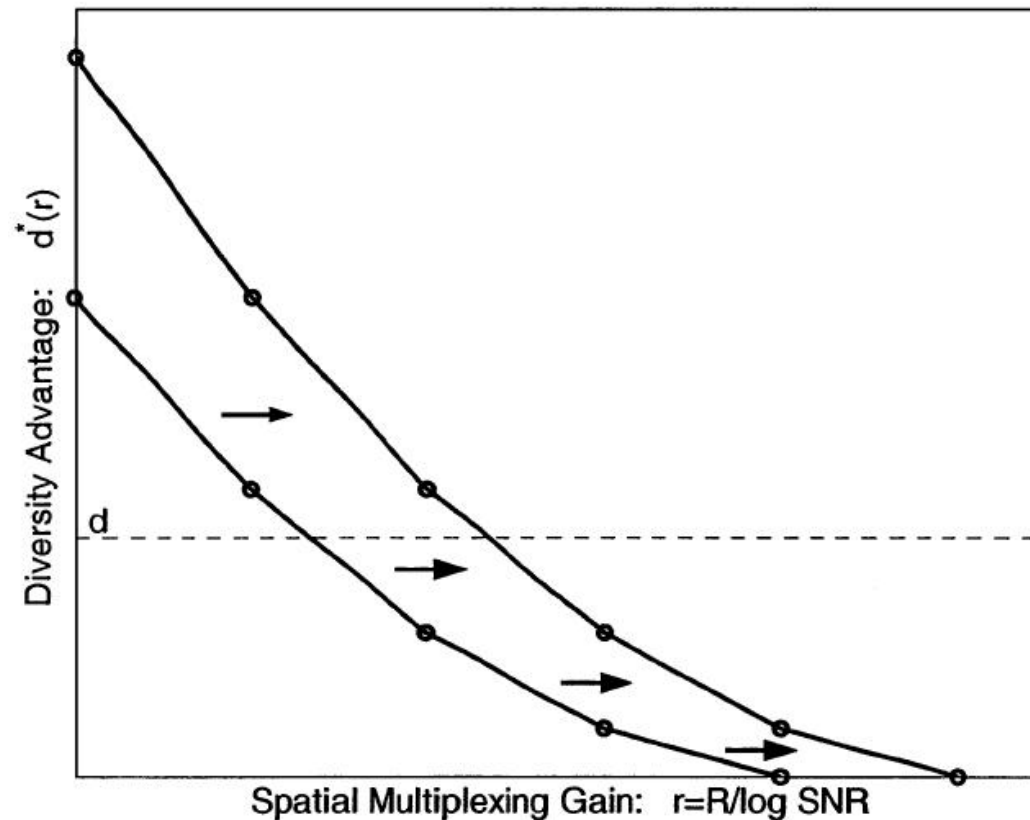


Figure 7: Adding one transmit and one receive antenna increases multiplexing gain by 1 at each diversity level. Curve shifted to the right by 1.

# Example: Two Practical Schemes

Repetition scheme vs. Alamouti scheme

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & 0 \\ 0 & \mathbf{x}_1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & -\mathbf{x}_2^* \\ \mathbf{x}_2 & \mathbf{x}_1^* \end{bmatrix}$$

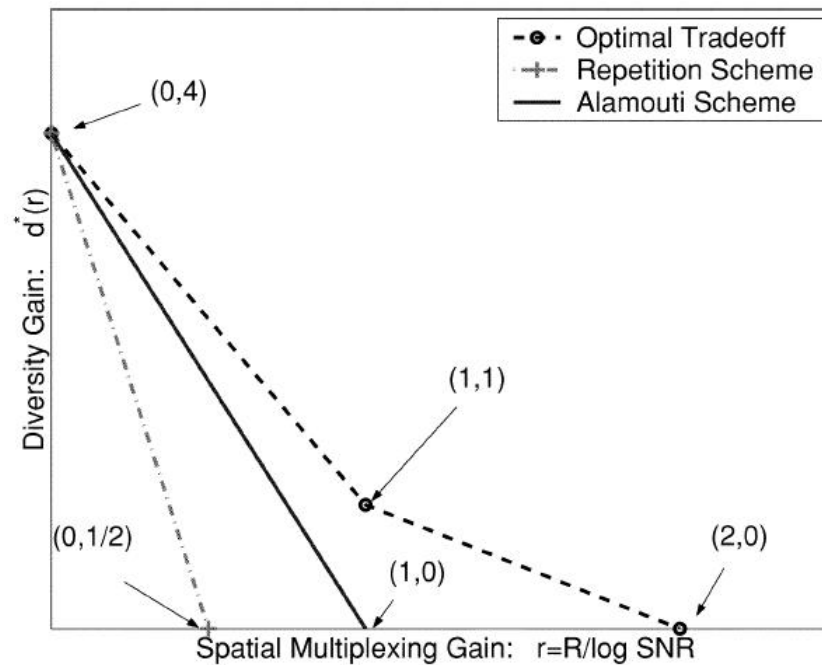


Figure 8: Diversity-Multiplexing tradeoff and comparison between two schemes.

## Beyond the examples

---

- If the performance of schemes only evaluated by maximal diversity gain  $d(0)$ , we cannot distinguish the performance of repetition scheme and Alamouti scheme.
- Different coding schemes may be designed for different goals.
  - Orthogonal Designs (e.g. Alamouti): Maximize the diversity gain.
  - V-BLAST: Maximize the spatial multiplexing gain.
- The diversity-multiplexing tradeoff provides a unified framework to make fair comparisons and helps us understand the characteristic of a particular scheme more completely.

# Beyond the examples

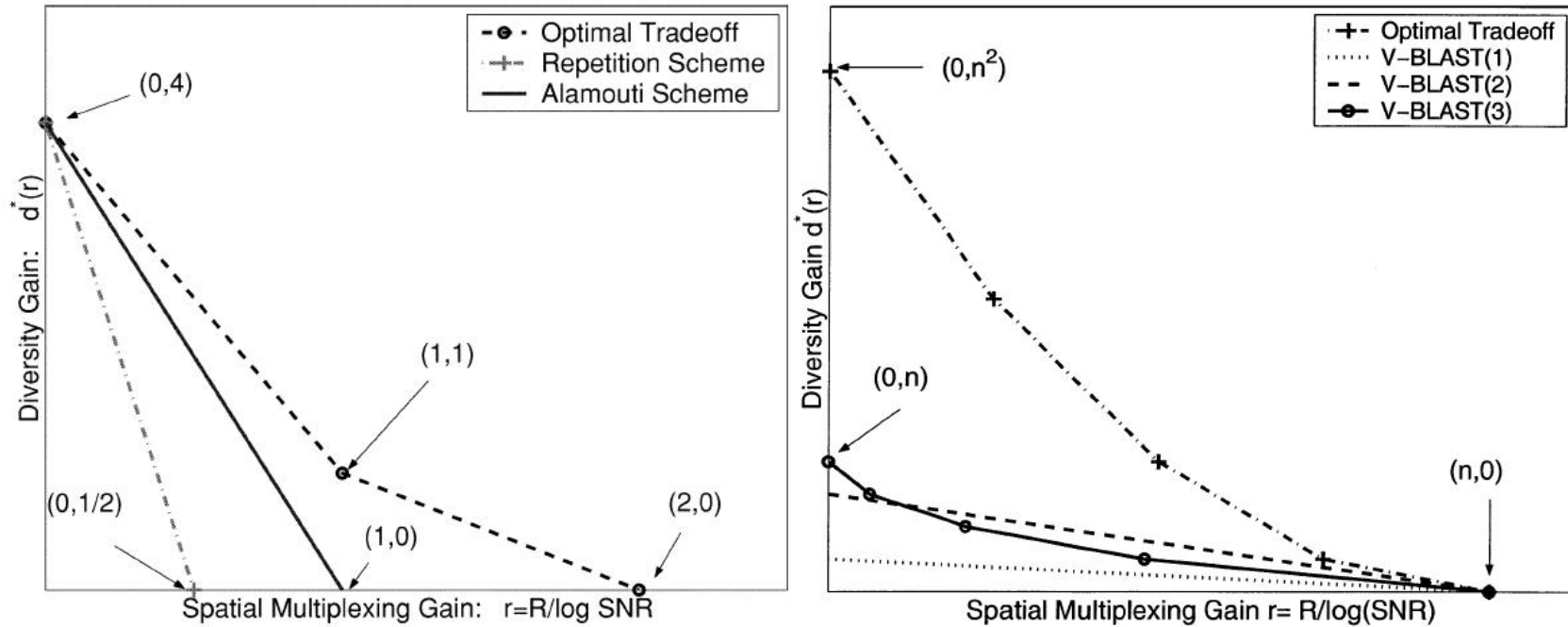


Figure 9: Orthogonal Designs vs. V-BLAST

## Extension to Multiple-Access Channels

---

- The discussion on diversity-multiplexing tradeoff can be extended to channels with multiple-access.
- Suppose we allow each user to have a diversity gain of  $d$ , then we can characterize the set of multiplexing gain tuples  $(r_1, r_2, \dots, r_K)$ .
- Theorem: If the block length  $l \geq Km + n - 1$ ,

$$\mathcal{R}(d) = \left\{ (r_1, r_2, \dots, r_K) : \sum_{s \in S} r_s \leq r_{|S|m,n}^*(d), \forall S \subseteq 1, \dots, K \right\}$$

where  $r_{|S|m,n}^*(\cdot)$  is the multiplexing-diversity tradeoff curve for a point-to-point channel with  $|S|m$  transmit and  $n$  receive antennas.

# Conclusion

---

- We introduce the basic idea of multiple-antenna channels.
- Derive the capacity of the channels with different setup. The difficulty in evaluating channel capacity. Look for more practical measures to compare coding scheme.
- Diversity and multiplexing gain. The fundamental tradeoff.
- Extension to multiple-access channel.

# References

---

- [1] I. E. Telatar, "Capacity of Multi-antenna Gaussian Channels," *Rm.2C-174*, Lucent Technologies, Bell Lab, 1997.
- [2] L. Zheng, D. Tse, "Diversity and Multiplexing: A Fundamental Tradeoff in Multiple-Antenna Channels," *IEEE Trans. on Information Theory*, Vol. 49, No. 5, May 2003.
- [3] D. Tse, P. Viswanath, L. Zheng, "Diversity-Multiplexing Tradeoff in Multiple-Access Channels," *IEEE Trans. on Information Theory*, Vol. 50, No. 9, Sep 2004.
- [4] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. New York: Wiley, 2006.
- [5] D. Tse, P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, May 2005.