The simplest canal surface is a surface swept by a moving circle. We describe here a procedure we call \textbf{gopher} that constructs the equation of such a surface. We have chosen this name since the procedure yields the parametric equations of a surface which depicts the boundary of the \textit{canal} that a gopher (of varying size) constructs as it digs its way underground. In other words, after the gopher digs its canal, paint the walls of the canal then, magically, remove all the dirt. What you are left with, in pure paint, is the resulting canal. The resulting surface is the surface of the canal swept away by the \textit{circle}, giving the \textit{equator} of the gopher (assuming that he/she is a bit plumpy). To give a moving circle, we need need give the parametric equations of the curve swept by its \textit{center}, its \textit{radius} as a function of time, and the plane that contains it. Let then

\begin{equation}
\{x[t], y[t], z[t]\} \quad \text{and} \quad R[t]
\end{equation}

respectively denote the \(x, y, z\)-coordinates of the center of the circle and its radius at time \(t\). It will also be convenient to let \(t\) vary from 0 to 1, refer to the curve \(\{x[t], y[t], z[t]\}\) as the \textit{directrix} and denote the moving circle at time \(t\) by \(C[t]\). This given, the procedure \textbf{gopher} returns the parametric equations of the corresponding canal surface. To simplify our task, we need to make some assumptions. There is no loss in assuming that \(C[t]\) lies in a plane that is perpendicular to the directrix at the point \(\{x[t], y[t], z[t]\}\). Let us assume first that the directrix is a planar curve. Later we will deal with canal surfaces with a non planar directrix.

If we assume that the directrix of our canal surface is in the \(y, z\)-plane we can then set \(x[t] = 0\) in (1) and we are only left with chosing \(y[t]\) and \(z[t]\). This given, if \(N[t]\) denotes a unit vector that is in the \(y, z\)-plane and is perpendicular to the directrix at the point \(P[t] = \{0, y[t], z[t]\}\) and we set \(B = \{1, 0, 0\}\) then the parametric equations of the circle \(C[t]\) may be written in vector form as

\begin{equation}
C[t] = P[t] + R[t](Cos[\alpha]N[t] + Sin[\alpha]B)
\end{equation}

To rewrite this in component form we need further ingredients. Recall that the unit vector tangent to the directrix at \(P[t] = (x[t], y[t], z[t])\) is given by the formula \(T = \hat{P}[t]/|\hat{P}[t]|\). Thus

\begin{equation}
T = \{ u_1[t] , u_2[t] , u_3[t] \}
\end{equation}

with

\begin{align*}
u_1(t) &= \frac{\dot{x}[t]}{\sqrt{\dot{x}[t]^2 + \dot{y}[t]^2 + \dot{z}[t]^2}}, \\
u_2(t) &= \frac{\dot{y}[t]}{\sqrt{\dot{x}[t]^2 + \dot{y}[t]^2 + \dot{z}[t]^2}}, \\
u_3(t) &= \frac{\dot{z}[t]}{\sqrt{\dot{x}[t]^2 + \dot{y}[t]^2 + \dot{z}[t]^2}}.
\end{align*}

Since here \(\dot{x}(t) = 0\), the vector \(T(t)\) is in the \(y, z\)-plane and thus \(N[t]\) may be computed by the cross product \(B \times T\). This gives

\begin{equation}
N[t] = \{ 0 , -u_3[t] , u_2[t] \}
\end{equation}

Putting (3) and (4) together we finally obtain that

\begin{equation}
N[t] = \begin{cases} 
0, & -\frac{\dot{z}[t]}{\sqrt{\dot{y}[t]^2 + \dot{z}[t]^2}}, \\
\frac{\dot{y}[t]}{\sqrt{\dot{y}[t]^2 + \dot{z}[t]^2}} & \end{cases}
\end{equation}

Making the substitutions \(P[t] = \{ 0, y[t], z[t] \}\) and \(B = \{ 1, 0, 0 \}\) and using (5) in (2) we derive that the parametric equations of the corresponding canal surface are

\begin{align*}
x[t, \alpha] &= R[t] Sin[\alpha] \\
y[t, \alpha] &= y[t] - \frac{R[t] Cos[\alpha] \dot{z}[t]}{\sqrt{\dot{y}[t]^2 + \dot{z}[t]^2}} \quad (\text{for} \ 0 \leq t \leq 1, \ 0 \leq \alpha \leq 2\pi) \\
z[t, \alpha] &= z[t] + \frac{R[t] Cos[\alpha] \dot{y}[t]}{\sqrt{\dot{y}[t]^2 + \dot{z}[t]^2}}
\end{align*}
We are now ready to wrap all this up. You start by writing a Mathematica procedure with heading gopher[dir_, R_] which upon a call of it, with dir given by a pair of functions \{y[t], z[t]\} and R given by a single function \(R(t)\), returns the triplet in (6). Note that once you have constructed the parametric equation of a surface, Mathematica will give you a very realistic graphic display by means of the ParametricPlot3D. You may include in your package the procedure

\[
PP3[surfaces\_] := \text{param}[surfaces, \{t, 0, 1\}, \{\text{alpha}, 0, 2\Pi\}, \text{AspectRatio} > 1, \text{Axes} > \text{True} ]/.\text{param} -> \text{ParametricPlot3D}
\]

which allows you to display several surfaces at the same time. You may also include the procedure

\[
VPO3[surfaces\_, a\_, b\_, c\_] := \text{param}[surfaces, \{t, 0, 1\}, \{\text{alpha}, 0, 2\Pi\}, \text{AspectRatio} > 1, \text{ViewPoint} > \{a, b, c\}, \text{Axes} > \text{True} ]/.\text{param} -> \text{ParametricPlot3D}
\]

which also allows you to choose the view point.

Finally we should mention that to obtain the most general canal surface, with directrix \(P(t) = \{x(t), y(t), z(t)\}\) we can still use formula (2) provided we set

\[
N(t) = \frac{\hat{T}(t)}{|T(t)|} \quad \text{and} \quad B(t) = T(t) \times N(t)
\]

The only problem in the general case is that at some points the derivative vector \(\hat{T}(t)\) may vanish and these formulas cannot be applied. However in these cases we may have to make use of higher order derivatives to construct \(N(t)\). Alternatively at such a point we can produce a normal vector \(N(t)\) as the cross product of a random vector by the tangent vector \(T(t)\).

Moreover, there is no compelling reason that the cross-section of our canal surfaces should be restricted to be a circle. We can simply use any Bezier curve as cross-section.

However, due to the fact that, our auxiliary display procedures assume that the parametric equations of our surfaces are of the form

\[
\{x(t, \alpha), y(t, \alpha), z(t, \alpha)\} \quad \text{(with } 0 \leq t \leq 1 \text{ and } 0 \leq \alpha \leq 2\pi)\]

then, if your desired cross-section is the Bezier curve

\[
f = \{a(t), b(t)\} \quad \text{(with } 0 \leq t \leq 1)\]

Then you must change \(t\) to \(\frac{\alpha}{2\pi}\) and use instead the parametric equations

\[
g = \{a(\frac{\alpha}{2\pi}), b(\frac{\alpha}{2\pi})\} \quad \text{(with } 0 \leq \alpha \leq 2\pi)\]

This can be simply accomplished by the Mathematica command

\[
g = f / . t -> \frac{\alpha}{2\Pi}
\]
In the above display I have placed the procedures that you may find in various forms in the MATHEMATICA "Canal Surfaces" notebook I posted in the course website. The “r” procedures use a random frame and the “s” procedures use a frame computed by differentiation. These frames are graphically better but they produce problems when the tangent vector has zero derivative at some points of the directrix.

When using either of the canal procedures the directrix should be the curve of your choice, the input parameter d must be your desired “stretching” function of t which controls by how much the size of the cross-section varies for $0 \leq t \leq 1$. If you do not want the size to change just take d to be a constant of your choice. Finally the input parameter cu is your desired cross-section. It should be a function of alpha as indicated above.

In this assignment you are to construct a surface of revolution of your choice and a canal surface of your choice. Be creative and have fun.

Here are a surface of revolution and two samples of canal surfaces