ASSIGNMENT III

ROTATION

We recall that the most general rotation of space is given by three angles α, β, γ. This rotation which we shall denote by $R(\alpha,\beta,\gamma)$ transforms the three basic coordinate vectors

$$\mathbf{e_1} = (1,0,0) \quad , \quad \mathbf{f_1} = (0,1,0) \quad , \quad \mathbf{g_1} = (0,0,1)$$

by three successive rotations of angles α, β, γ according to the following procedure

(1) First a rotation of angle α around $\mathbf{g_1}$ keeping $\mathbf{g_1}$ fixed, yielding three new coordinate vectors

$$\mathbf{e_2} = cos\alpha \;\; \mathbf{e_1} + sin\alpha \;\; \mathbf{f_1} \quad , \qquad \mathbf{f_2} = -sin\alpha \;\; \mathbf{e_1} + cos\alpha \;\; \mathbf{f_1} \quad , \qquad \mathbf{g_2} = \mathbf{g_1} \tag{α}$$

(2) Then a rotation of angle β around $\mathbf{e_2}$ keeping $\mathbf{e_2}$ fixed, yielding three new coordinate vectors

$$\mathbf{e_3} = \mathbf{e_2} \quad , \qquad \mathbf{f_3} = cos\beta \;\; \mathbf{f_2} + sin\beta \;\; \mathbf{g_2} \quad , \qquad \mathbf{g_3} = -sin\beta \;\; \mathbf{f_2} + cos\beta \;\; \mathbf{g_2} \tag{β}$$

(2) Finally a rotation of angle γ around $\mathbf{g_3}$ keeping $\mathbf{g_3}$ fixed, yielding three new coordinate vectors

$$\mathbf{e_4} = cos\alpha \;\; \mathbf{e_3} + sin\alpha \;\; \mathbf{f_3} \quad , \qquad \mathbf{f_4} = -sin\alpha \;\; \mathbf{e_3} + cos\alpha \;\; \mathbf{f_3} \quad , \qquad \mathbf{g_4} = \mathbf{g_3} \tag{γ}$$

Carrying out the substitutions given in (α), (β) and (γ), a simple computation (that may be checked with MATHEMATICA), yields that

$$\begin{array}{rclccccccc}
R(\alpha,\beta,\gamma)\;\mathbf{e_1} & = & (& cos\gamma\;cos\alpha - \;sin\alpha\;cos\beta\;sin\gamma & , & sin\alpha\;cos\gamma + \;cos\alpha\;cos\beta\;sin\gamma & , & sin\beta\;sin\gamma &) \\
R(\alpha,\beta,\gamma)\;\mathbf{f_1} & = & (& -\;cos\alpha\;sin\gamma - \;sin\alpha\;cos\beta\;cos\gamma & , & -\;sin\alpha\;sin\gamma + \;cos\alpha\;cos\beta\;cos\gamma & , & sin\beta\;cos\gamma &) \\
R(\alpha,\beta,\gamma)\;\mathbf{g_1} & = & (& sin\alpha\;sin\beta & , & -\;cos\alpha\;sin\beta & , & cos\beta &)
\end{array}$$

If we take a point P in space with coordinates (x,y,z) its image by the rotation $R(\alpha,\beta,\gamma)$, is by definition, the point

$$P' = (x',y',z') = R(\alpha,\beta,\gamma)\;P$$

whose coordinates are given by the equation

$$R(\alpha,\beta,\gamma)\;P \;\; = x\;R(\alpha,\beta,\gamma)\;\;\mathbf{e_1} \;+\; y\;R(\alpha,\beta,\gamma)\;\;\mathbf{f_1} \;+\; z\;R(\alpha,\beta,\gamma)\;\;\mathbf{g_1} \;\; . \tag{1}$$

For convenience, let $R = \|R[i;j]\|$ be the matrix

$$R = \begin{pmatrix} cos\gamma\;cos\alpha - \;sin\alpha\;cos\beta\;sin\gamma & -\;cos\alpha\;sin\gamma - \;sin\alpha\;cos\beta\;cos\gamma & sin\alpha\;sin\beta \\ sin\alpha\;cos\gamma + \;cos\alpha\;cos\beta\;sin\gamma & -\;sin\alpha\;sin\gamma + \;cos\alpha\;cos\beta\;cos\gamma & -\;cos\alpha\;sin\beta \\ sin\beta\;sin\gamma & sin\beta\;cos\gamma & cos\beta \end{pmatrix}$$

This given, it is not difficult to show that (1) is equivalent to the equations

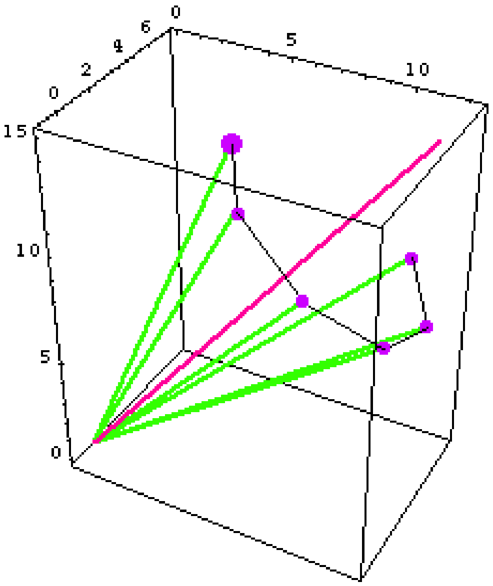
$$\begin{array}{rcl} x' & = & R_{11}\,x \;+\; R_{12}\,y \;+\; R_{13}\,z \\ y' & = & R_{21}\,x \;+\; R_{22}\,y \;+\; R_{23}\,z \\ z' & = & R_{31}\,x \;+\; R_{32}\,y \;+\; R_{33}\,z \end{array} \tag{2}$$

Note that, using matrix notation, these equations can simply be written in the form $v' = R\,v$ where v and v' respectively denote the column vectors with components x, y, z and x', y', z'.

The rotation matrix we have presented above is customarily used in Astronomy. Its advantage lies in that it contains the smallest number of parameters. But it has the disadvantage of not exhibiting the fundamental fact that every rigid motion of space which as a fixed point has also a fixed line. This means that the motion can always be achieved by a single rotation around a fixed line.

We shall present next another way of moving objects, which is based on this principle. You may use this alternate way in carrying out the assignment.

The idea is to chose first a vector **D** giving the direction and orientation of the fixed line **L**. Then choose the angle α by which we wish to rotate † In the example below we exhibit **D** as a dark line, a given point **P** by a large disk and, by smaller disks, the successive images of **P** as we carry 5 successive rotations of angle $2\pi/8$.



Suppose that $\mathbf{D} = \{A, B, C\}$ and we want to rotate the point $\mathbf{P} = \{x, y, z\}$ around **L** by the angle α, then the equation giving the point $\mathbf{P'} = \{x', y', z'\}$ which is the image of **P** under this rotation, can be can be obtained as follows. To begin with we compute the unit vector $\mathbf{d} = \{u, v, w\}$ in the direction of **L** by setting

$$u = \frac{A}{\sqrt{A^2+B^2+C^2}}\ , \qquad v = \frac{B}{\sqrt{A^2+B^2+C^2}}\ , \qquad w = \frac{C}{\sqrt{A^2+B^2+C^2}}\ ,$$

Next, we compute the auxiliary vectors:

$$\begin{aligned} \mathbf{P_1} &= (\mathbf{d}, \mathbf{P})\ \mathbf{d} && \text{(projection of } \mathbf{P} \text{ along } \mathbf{L}\text{)} \\ \mathbf{P_2} &= \mathbf{P} - \mathbf{P_1} && \text{(the portion of } \mathbf{P} \text{ perpendicular to } \mathbf{L}\text{)} \\ \mathbf{N} &= \mathbf{d} \times \mathbf{P_2} && \text{(the vector obtained by rotating } \mathbf{P_2} \text{ by } 90^o \text{ around } \mathbf{L}\text{)} \end{aligned}$$

This done we simply have

$$\mathbf{P'} = \mathbf{P_1} + Cos[\alpha]\,\mathbf{P_2} + Sin[\alpha]\,\mathbf{N}\ .$$

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† Counterclockwise as seen by a person with feet at the origin of **D** and head on the tip of **D**

To implement this write a procedure with heading

$$\mathbf{ROTPT}[\mathbf{P}_{-},\mathbf{D}_{-},\alpha_{-}]$$

which returns $\mathbf{P'}$ as indicated above, taking account that $(\mathbf{d},\mathbf{P})$ and $\mathbf{d}\ \times\ \mathbf{P_2}$ denote "*scalar*" and "*vector*" products respectively. More precisely,

$$(\mathbf{d},\mathbf{P})\ =\ u\,x\ +\ v\,y\ +\ w\,z$$

and, if $\mathbf{P_2}=\{x_2,y_2,z_2\}$ then

$$\mathbf{d}\ \times\ \mathbf{P_2}\ =\ \{\,v\,z_2-w\,y_2\ ,\ \ w\,x_2-u\,z_2\ ,\ \ u\,y_2-v\,x_2\,\}$$

We shall learn how to display solid objects in the screen by working with objects consisting of a certain number of solid polyhedra. A polyhedron can and should be presented as a list of its planar faces. In turn, each face is presented by giving the (x,y,z) coordinates of the successive vertices of its polygonal boundary arranged in the so called *outer counterclockwise order.* The latter being the order we encounter them if we run around the boundary counterclock-wise relative to the outer normal to the face. (†) For instance, a cube with center in the origin and semiside length 1, shall be presented in MATHEMATICA as the concatenation of its six faces. That is we set

$$CUBE=\{F1,F2,F3,F4,F5,F6\} \tag{3}$$

with the top face $F1$ presented as

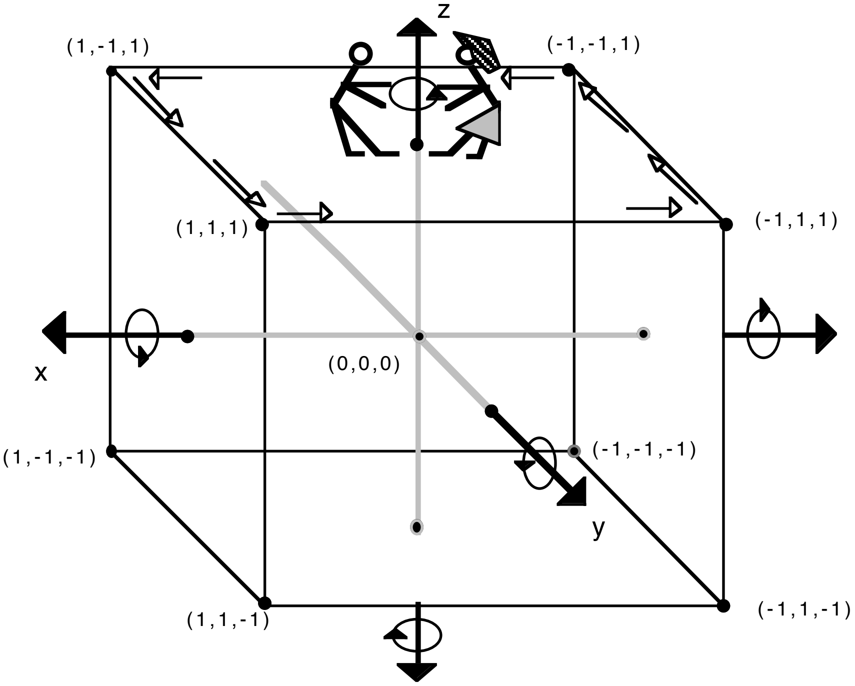
$$F1=\{\{1,1,1\},\{-1,1,1\},\{-1,-1,1\},\{1,-1,1\},\{1,1,1\}\} \tag{4}$$

With this conventions the remaining faces are given by the sequences

$$\begin{aligned}
&F6=\{\{1,1,-1\},\{1,-1,-1\},\{-1,-1,-1\},\{-1,1,-1\},\{1,1,-1\}\} \qquad (bottom)\\
&F2=\{\{1,1,1\},\{1,1,-1\},\{-1,1,-1\},\{-1,1,1\},\{1,1,1\}\} \qquad (front)\\
&F4=\{\{1,-1,1\},\{-1,-1,1\},\{-1,-1,-1\},\{1,-1,-1\},\{1,-1,1\}\}\} \qquad (back)\\
&F5=\{\{1,1,1\},\{1,-1,1\},\{1,-1,-1\},\{1,1,-1\},\{1,1,1\},\} \qquad (left)\\
&F3=\{\{-1,1,1\},\{-1,1,-1\},\{-1,-1,-1\},\{-1,-1,1\},\{-1,1,1\},\} \qquad (right)
\end{aligned} \tag{5}$$

(†) The reason for doing this will be apparent when we carry out our next assignment.

You should check that the above data are in agreement with the picture below:



Write a MATHEMATICA procedure, with heading

$$\mathbf{ROTATE}[\mathbf{object}_{-},\mathbf{angles}_{-}]$$

where **object** is a list of faces of a polyhedron (such as that defined by (3),(4),(5)) and **angles** gives the three angles of rotation. This procedure should rotate the object by the rotation $R(\alpha,\beta,\gamma)$ and then display the rotated object. The rotation can be carried out by applying the linear transformation in (2) to the coordinates (x,y,z) of each of the vertices of each of the faces of the object. This can be done by a single MATHEMATICA matrix multiplication command. Thereafter, the procedure should transform the resulting sequencve of faces into a sequence of graphic commands of the form

$$\mathbf{Polygon}[\{\{x_1,y_1\},\{x_2,y_2\},\{x_3,y_3\},\ldots\}]$$

one for each rotated face, with $\{x_1,y_1\},\{x_2,y_2\},\{x_3,y_3\},\ldots$ giving the (x,y) coordinates of its successive vertices. It might be good, in order for you to have an idea of what exactly happened to the object, to have your procedure depict the coordinate axes. Construct your favorite convex object according to the rules given above and exhibit a few rotated images of it.